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Optimal Runge-Kutta Schemes for High-order Spatial and Temporal Discretizations

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2015 AIAA SciTech
June 23, 2015

Outline

- **Introduction**
- **Governing Equations**
  - Spatial Discretizations
  - Temporal Discretizations
- **Von Neumann Analysis (VNA)**
- **Computational Results**
  - One-dimensional Wave
  - Three-dimensional Vortex
- **Conclusions and Future Work**

Introduction

- **High-order in space is now commonplace**
- **High-order in time… not so much…**
- **Is this sufficient? Is high-order in time needed?**
- ***Limiting Fact:* There are no *A*-stable backward-difference formula (BDF) methods with > *2nd* -order accuracy**
- **Thus, multistage methods, like Runge-Kutta (RK) methods, must be used for *3rd*- and higher-order**
- **Explicit RK methods are not amenable to stiff problems**

***Objective:* To find optimal diagonally-implicit Runge-Kutta time integrators for use with high-order spatial discretizations**

Governing Equations

- **Dual Time Stepping:**

$$\frac{\partial \mathbf{Q}}{\partial \tau} + \frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}_i}{\partial x_i} = \frac{\partial \mathbf{V}_i}{\partial x_i} + \mathbf{H} \qquad \mathbf{Q} = \begin{bmatrix} \rho & \rho u_i & \rho e_0 \end{bmatrix}^T$$

$$\mathbf{F}_i = \begin{bmatrix} \rho u_i & \rho u_i u_j + p\delta_{ij} & u_i \rho h_0 \end{bmatrix}^T \text{ where } h_0 = e_0 + \tfrac{p}{\rho}$$

- **Quasi-linear Form:**

$$\frac{\partial \mathbf{Q}}{\partial \tau} + \frac{\partial \mathbf{Q}}{\partial t} + \underline{\mathbf{A}}\frac{\partial \mathbf{Q}}{\partial x_i} = \frac{\partial \mathbf{V}_i}{\partial x_i} + \mathbf{H} \qquad \underline{\mathbf{A}} = \tfrac{\partial \mathbf{F}_i}{\partial \mathbf{Q}} = \underline{\mathbf{M \Lambda M}}^{-1}$$

$$\underline{\mathbf{\Lambda}} = diag\left\{u_i + c, u_i, u_i - c\right\}$$

- **Residual Form:**

$$\frac{\partial \mathbf{Q}}{\partial \tau} + \frac{\partial \mathbf{Q}}{\partial t} + \mathbf{R}_s\left(\mathbf{Q}\right) = 0 \quad where \quad \mathbf{R}_s = \frac{\partial \mathbf{F}_i}{\partial x_i} - \frac{\partial \mathbf{V}_i}{\partial x_i} - \mathbf{H}$$

Spatial Discretizations

- **Central Differences with added artificial dissipation**
- **Central differences:**

$$\left.\frac{\partial \Upsilon_j}{\partial x_i}\right|_{II} = \frac{\Upsilon_{j+1} - \Upsilon_{j-1}}{2\Delta x_i}$$

$$\left.\frac{\partial \Upsilon_j}{\partial x_i}\right|_{IV} = \frac{-\Upsilon_{j+2} + 8\Upsilon_{j+1} - 8\Upsilon_{j-1} + \Upsilon_{j-2}}{12\Delta x_i}$$

$$\left.\frac{\partial \Upsilon_j}{\partial x_i}\right|_{VI} = \frac{\Upsilon_{j+3} - 9\Upsilon_{j+2} + 45\Upsilon_{j+1} - 45\Upsilon_{j-1} + 9\Upsilon_{j-2} - \Upsilon_{j-3}}{60\Delta x_i}$$

where Υ could be $\mathbf{F}_i$ or $\mathbf{Q}$ depending on the form of the equations

- **Scalar artificial dissipation:**

$$\mathbf{R}_s = \frac{\partial \mathbf{F}_i}{\partial x_i} - \varepsilon_\eta \parallel \lambda \parallel \frac{\partial^\eta \mathbf{Q}}{\partial x_i^\eta} - \frac{\partial \mathbf{V}_i}{\partial x_i} - \mathbf{H}$$

where η is even and one more than the order of accuracy

$$\parallel \lambda \parallel = |u_i| + c \qquad \varepsilon_{II} = \frac{\Delta x_i}{2}, \quad \varepsilon_{IV} = -\frac{\Delta x_i^3}{12}, \quad \varepsilon_{VI} = \frac{\Delta x_i^5}{60}.$$

Temporal Discretizations

- **Runge-Kutta Methods:**

$$
\begin{array}{c|cccccc}
c_1 & a_{11} & a_{12} & a_{13} & \cdots & a_{1(s-1)} & a_{1s} \\
c_2 & a_{21} & a_{22} & a_{23} & \cdots & a_{2(s-1)} & a_{2s} \\
c_3 & a_{31} & a_{32} & a_{33} & \cdots & a_{3(s-1)} & a_{3s} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_{s-1} & a_{(s-1)1} & a_{(s-1)2} & a_{(s-1)3} & \cdots & a_{(s-1)(s-1)} & a_{(s-1)s} \\
c_s & a_{s1} & a_{s2} & a_s3 & \cdots & a_{s(s-1)} & a_{ss} \\
\hline
 & b_1 & b_2 & b_3 & \cdots & b_{s-1} & b_s \\
 & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \cdots & \hat{b}_{s-1} & \hat{b}_s
\end{array}
$$

$$t^k = t^n + c_k \Delta t \qquad \mathbf{Q}^k = \mathbf{Q}^n - \Delta t \sum_{j=1}^{s} a_{kj} \mathbf{R}_s^j(\mathbf{Q}^j) \qquad k = 1, 2, \ldots, s$$

$$\mathbf{Q}^{n+1} = \mathbf{Q}^n - \Delta t \sum_{j=1}^{s} b_j \mathbf{R}_s^j(\mathbf{Q}^j) \qquad \hat{\mathbf{Q}}^{n+1} = \mathbf{Q}^n - \Delta t \sum_{j=1}^{s} \hat{b}_j \mathbf{R}_s^j(\mathbf{Q}^j)$$

$$\epsilon^{n+1} = \mathbf{Q}^{n+1} - \hat{\mathbf{Q}}^{n+1}$$

ESDIRK Methods

- **E**xplicit first stage **S**ingly-**D**iagonally **I**mplicit **R**unge-**K**utta
  - Stiffly accurate
  - Second-order stage accuracy
  - FSAL – **F**irst is the **S**ame **A**s **L**ast

$$
\begin{array}{c|cccccc}
c_1 = 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
c_2 & a_{21} & \lambda & 0 & \dots & 0 & 0 \\
c_3 & a_{31} & a_{32} & \lambda & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_{s-1} & a_{(s-1)1} & a_{(s-1)2} & a_{(s-1)3} & \dots & \lambda & 0 \\
c_s = 1 & b_1 & b_2 & b_3 & \dots & b_{s-1} & \lambda \\
\hline
 & b_1 & b_2 & b_3 & \dots & b_{s-1} & \lambda \\
 & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \dots & \hat{b}_{s-1} & \hat{b}_s
\end{array}
$$

ESDIRK3 and 4

$$
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
\frac{1767732205903}{2027836641118} & \frac{1767732205903}{4055673282236} & \frac{1767732205903}{4055673282236} & 0 & 0 \\
\frac{3}{5} & \frac{2746238789719}{10658868560708} & -\frac{640167445237}{6845629431997} & \frac{1767732205903}{4055673282236} & 0 \\
1 & \frac{1471266399579}{7840856788654} & -\frac{4482444167858}{7529755066697} & \frac{11266239266428}{11593286722821} & \frac{1767732205903}{4055673282236} \\
\hline
 & \frac{1471266399579}{7840856788654} & -\frac{4482444167858}{7529755066697} & \frac{11266239266428}{11593286722821} & \frac{1767732205903}{4055673282236}
\end{array}
$$

Implicit, Third-order ESDIRK3

$$
\begin{array}{c|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\
\frac{83}{250} & \frac{8611}{62500} & -\frac{1743}{31250} & \frac{1}{4} & 0 & 0 & 0 \\
\frac{31}{50} & \frac{5012029}{34652500} & -\frac{654441}{2922500} & \frac{174375}{388108} & \frac{1}{4} & 0 & 0 \\
\frac{17}{20} & \frac{15267082809}{155376265600} & -\frac{71443401}{120774400} & \frac{730878875}{902184768} & \frac{2285395}{8070912} & \frac{1}{4} & 0 \\
1 & \frac{82889}{524892} & 0 & \frac{15625}{83664} & \frac{69875}{102672} & -\frac{2260}{8211} & \frac{1}{4} \\
\hline
 & \frac{82889}{524892} & 0 & \frac{15625}{83664} & \frac{69875}{102672} & -\frac{2260}{8211} & \frac{1}{4}
\end{array}
$$

Implicit, Fourth-order ESDIRK4

ESDIRK5

$$
\begin{array}{c|cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{41}{100} & \frac{41}{200} & \frac{41}{200} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2935347310677}{11292855782101} & \frac{41}{400} & -\frac{567603406766}{11931857230679} & \frac{41}{200} & 0 & 0 & 0 & 0 & 0 \\
\frac{1426016391358}{7196633302097} & \frac{683785636431}{9252920307686} & 0 & -\frac{110385047103}{1367015193373} & \frac{41}{200} & 0 & 0 & 0 & 0 \\
\frac{92}{100} & \frac{3016520224154}{10081342136671} & 0 & \frac{30586259806659}{12414158314087} & -\frac{22760509404356}{11113319521817} & \frac{41}{200} & 0 & 0 & 0 \\
\frac{24}{100} & \frac{218866479029}{1489978393911} & 0 & \frac{638256894668}{5436446318841} & -\frac{1179710474555}{5321154724896} & -\frac{60928119172}{8023461067671} & \frac{41}{200} & 0 & 0 \\
\frac{3}{5} & \frac{1020004230633}{5715676835656} & 0 & \frac{25762820946817}{25263940353407} & -\frac{2161375909145}{9755907335909} & -\frac{211217309593}{5846859502534} & -\frac{4269925059573}{7827059040749} & \frac{41}{200} & 0 \\
1 & -\frac{872700587467}{9133579230613} & 0 & 0 & \frac{22348218063261}{9555858737531} & -\frac{1143369518992}{8141816002931} & -\frac{39379526789629}{19018526304540} & \frac{32727382324388}{42900044865799} & \frac{41}{200} \\
\hline
 & -\frac{872700587467}{9133579230613} & 0 & 0 & \frac{22348218063261}{9555858737531} & -\frac{1143369518992}{8141816002931} & -\frac{39379526789629}{19018526304540} & \frac{32727382324388}{42900044865799} & \frac{41}{200}
\end{array}
$$

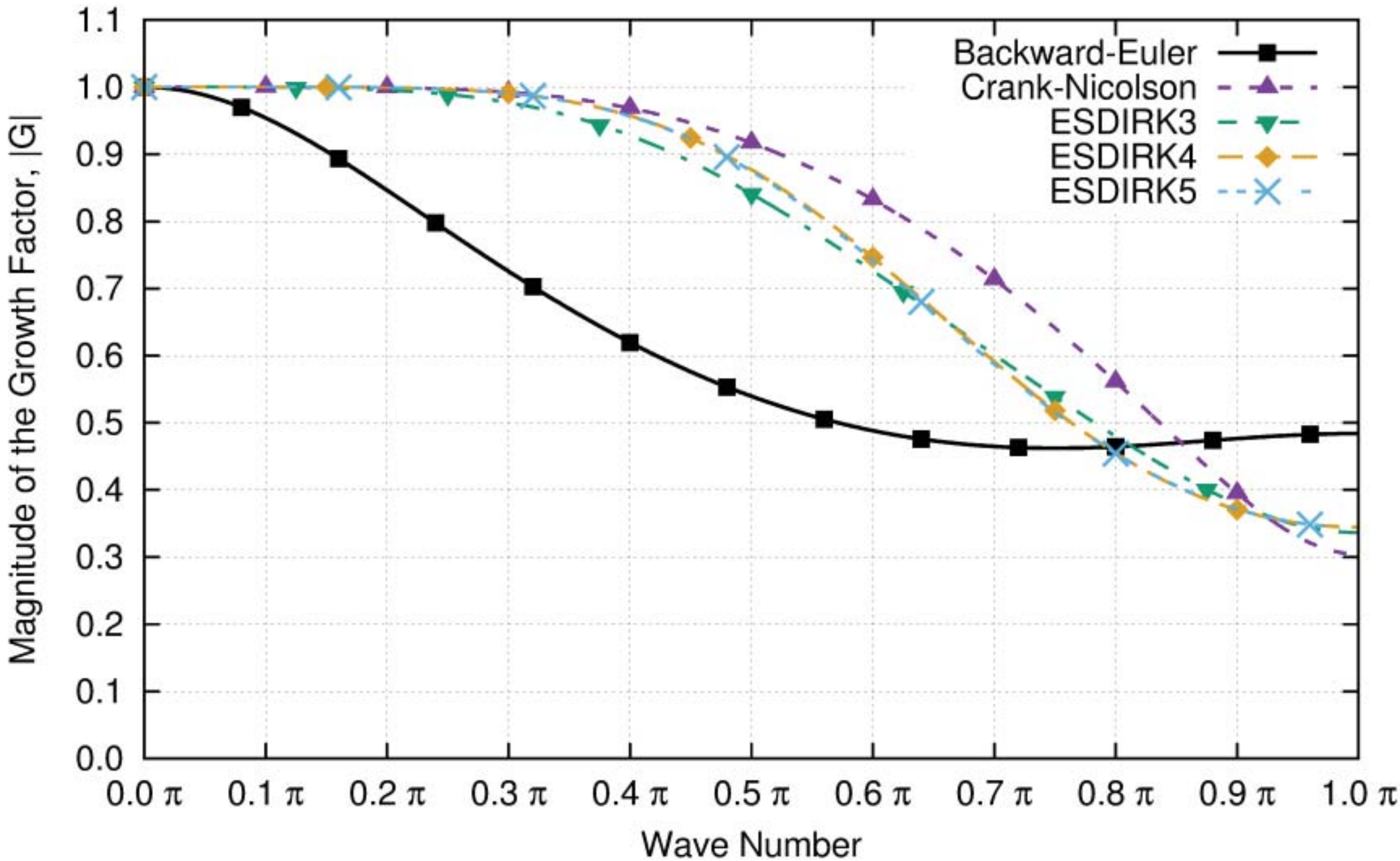
Implicit, Fifth-order ESDIRK5

The single biggest drawback of using these schemes is typing them out!

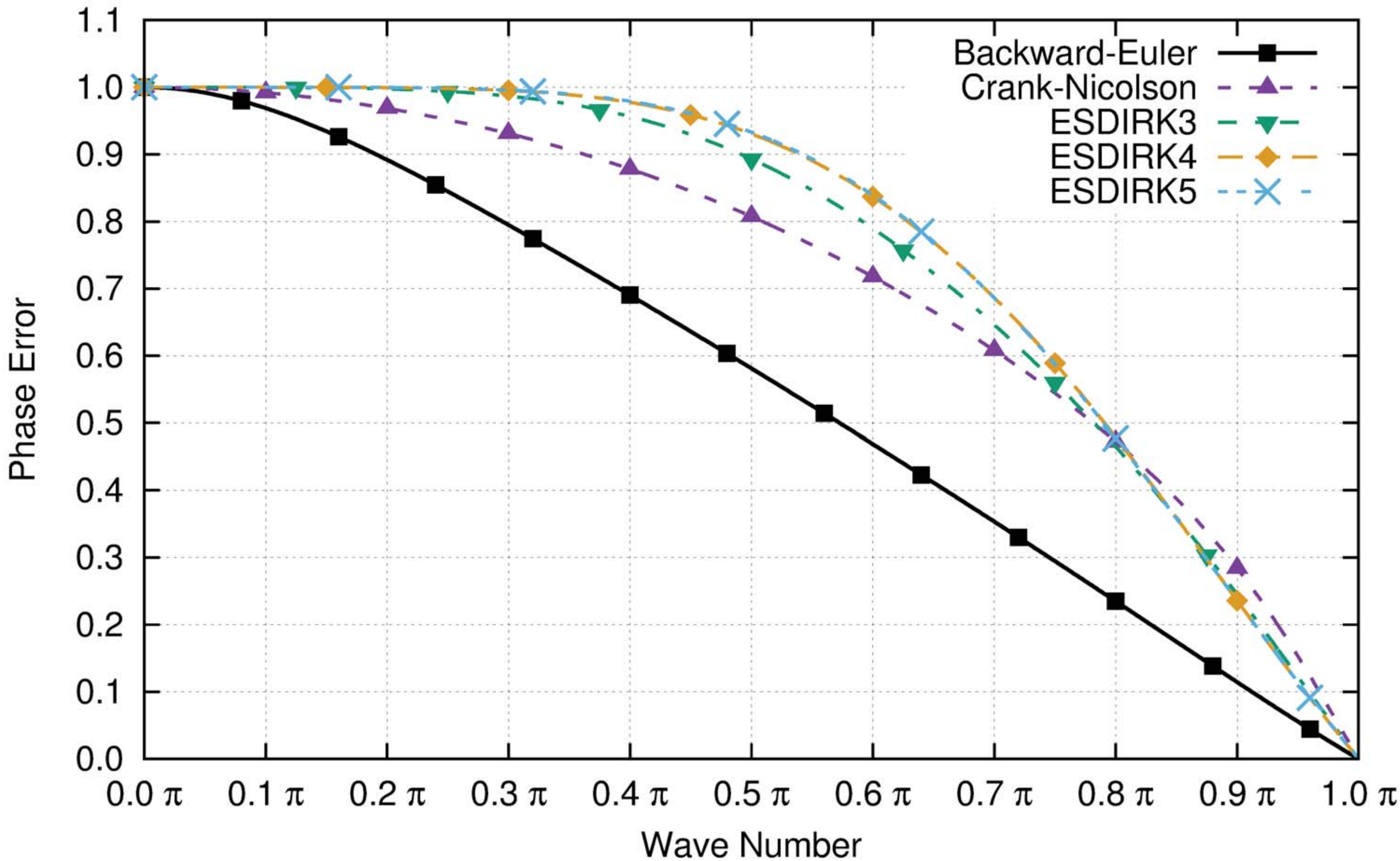
Von Neumann Analysis

- **Often used to study stability of schemes**
- **Von Neumann analysis is used to compare schemes for accuracy**
  - Dissipation error
  - Dispersion error
- **Assumes linear, periodic problems**
- **VNA theory and more results are in the associated paper**

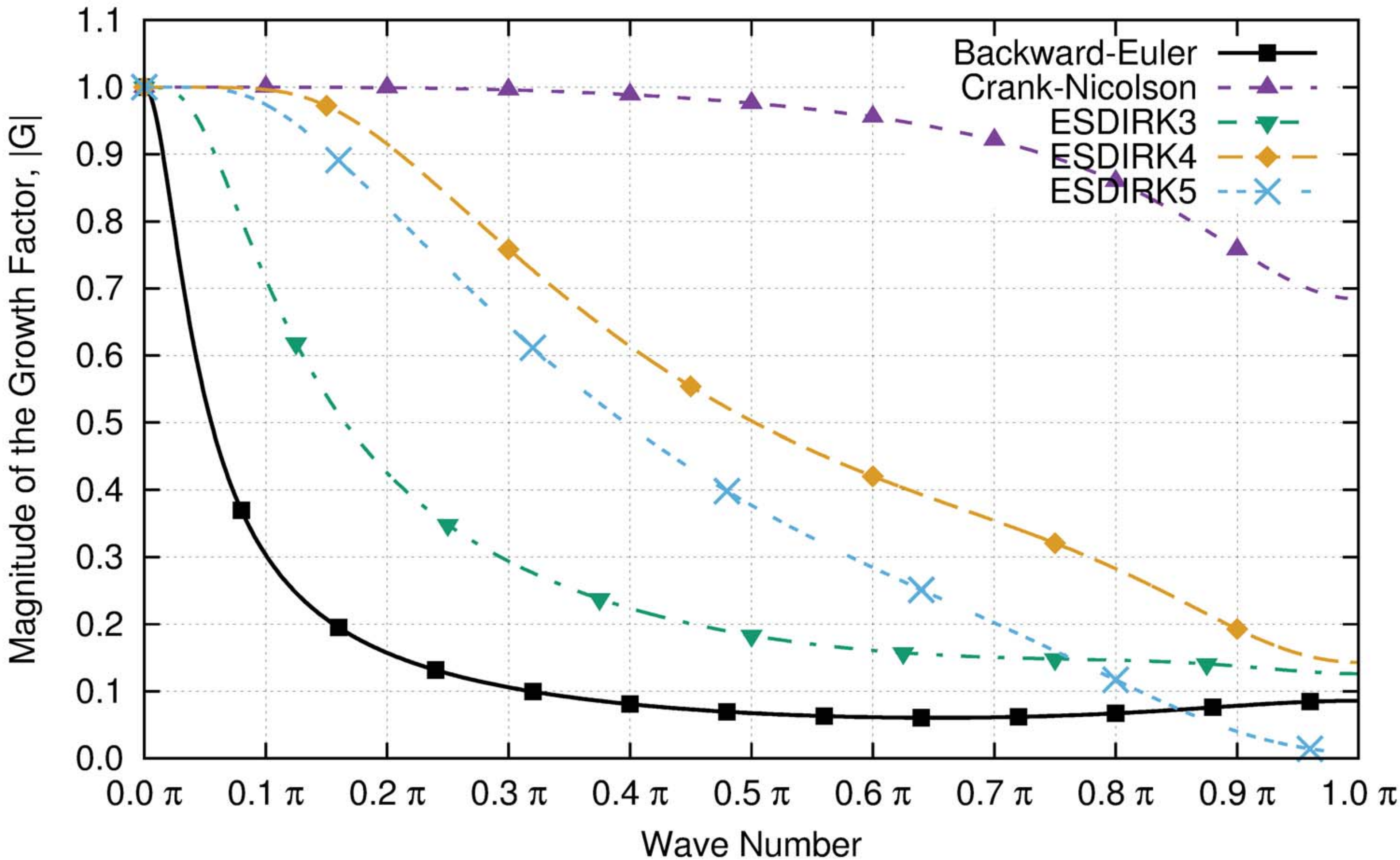
Dissipation, *CFL = 1.0*



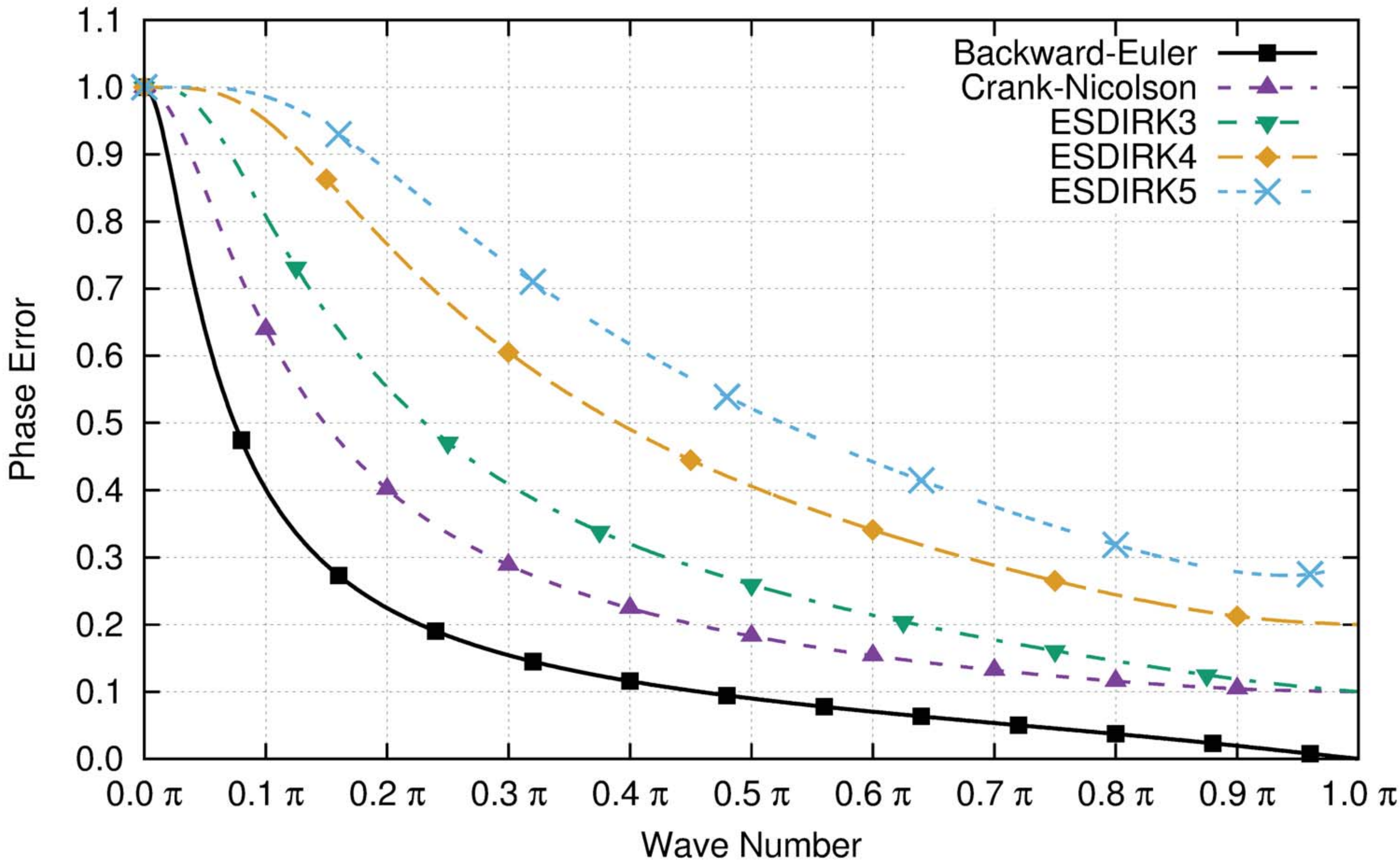
Dispersion, *CFL = 1.0*



Dissipation, *CFL* = 10.0



Dispersion, *CFL* = 10.0



1-D Acoustic Wave

- **Unperturbed Mach number of *0.5***

$$
\begin{aligned}
\rho_\infty &= 8.7077 \times 10^{-1} \frac{kg}{m^3} \\
\rho u_\infty &= 1.7458 \times 10^{2} \frac{kg}{m^2 \cdot s} \\
T_\infty &= 400K \\
R_\infty &= 2.871 \times 10^{2} \frac{J}{kg \cdot K} \\
\gamma &= 1.4
\end{aligned}
$$

- **Perturbation wave - 20 points per wave resolution**

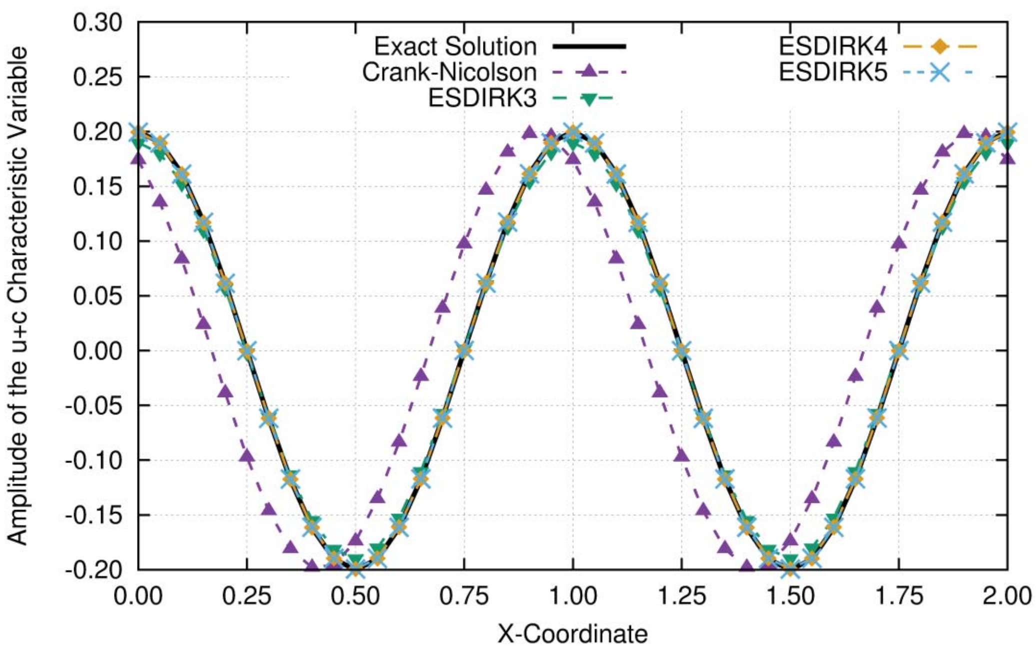
$$
\begin{aligned}
Q_o &= Q_\infty + M\delta\hat{Q}_{u,u\pm c} \\
\delta\hat{Q}_{u,u\pm c} &= \hat{\delta} \cdot \cos\left(kx\right)
\end{aligned}
$$

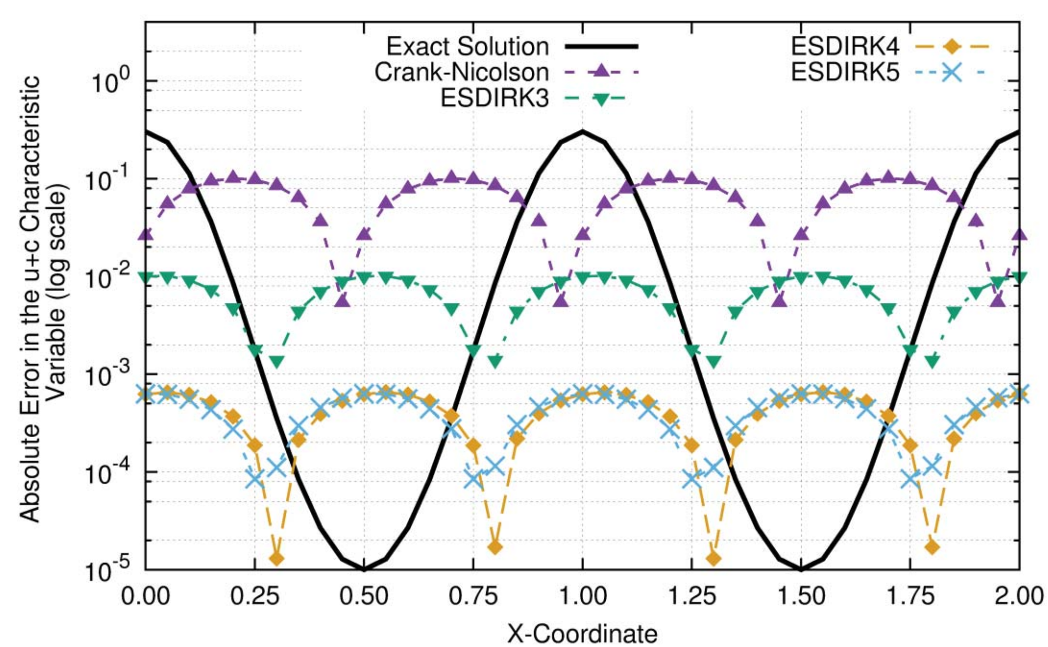
$$\text{where } \hat{\delta} = 0.01$$

- **More results in the paper**

1-D, *CFL* = *1.0*, 10 Periods

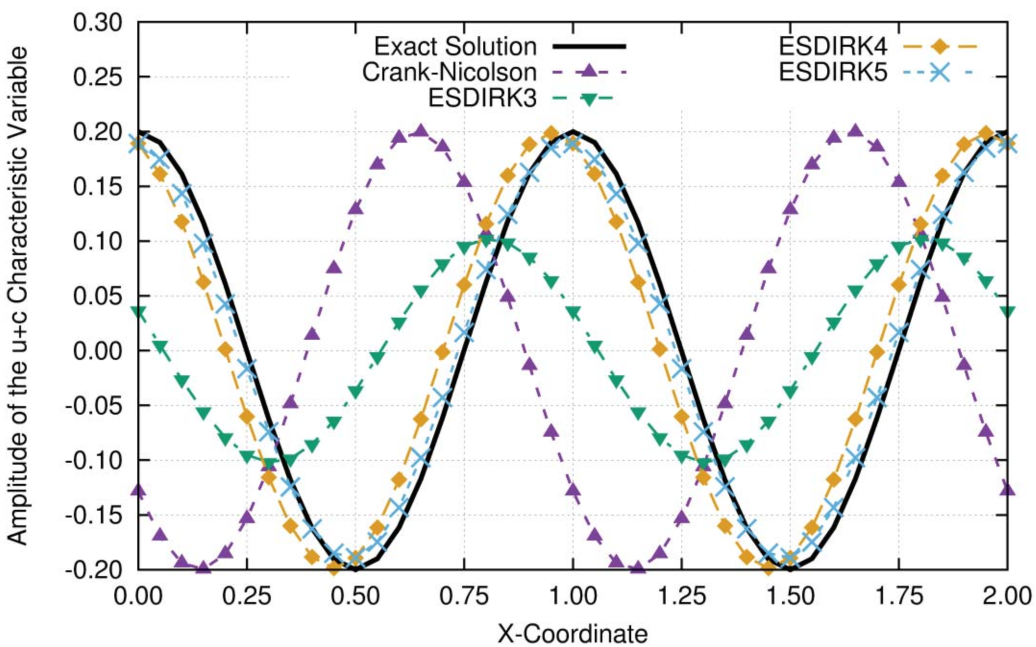
	Dissipation Error		Dispersion Error	
Scheme	VNA	Simulation	VNA	Simulation
Crank-N-colson	3.05×10^{-3}	1.00×10^{-2}	8.11×10^{-2}	8.11×10^{-2}
ESDIRK3	5.02×10^{-2}	5.02×10^{-2}	1.51×10^{-3}	1.53×10^{-3}
ESDIRK4	3.13×10^{-3}	3.13×10^{-3}	1.50×10^{-4}	1.58×10^{-4}
ESDIRK5	3.14×10^{-3}	3.14×10^{-3}	6.78×10^{-5}	6.90×10^{-5}

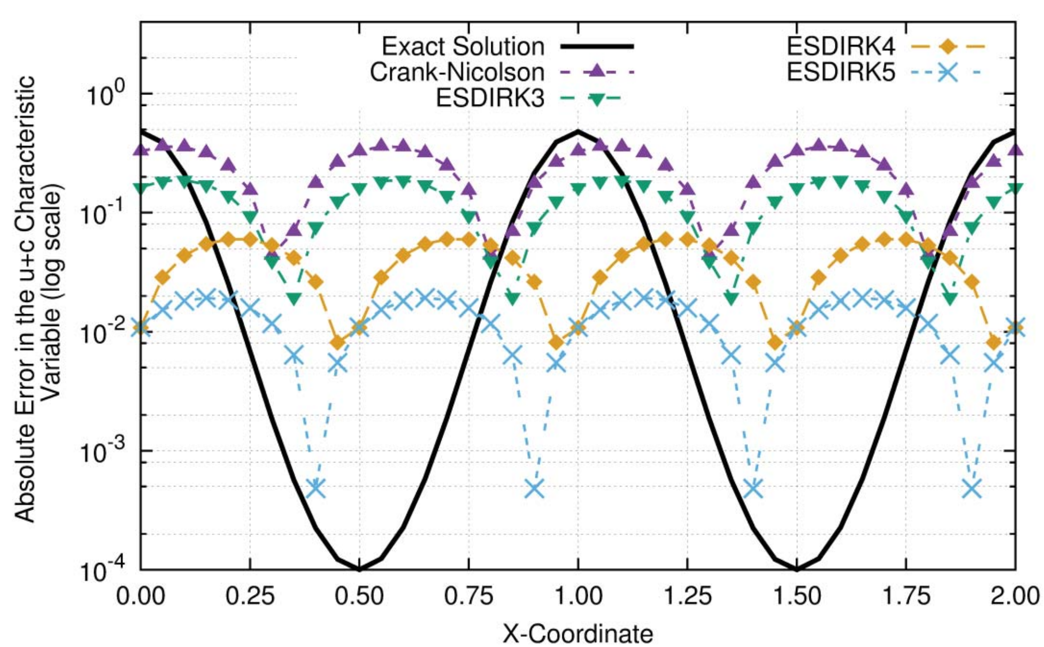




1-D, *CFL* = *10.0*, 1 Period

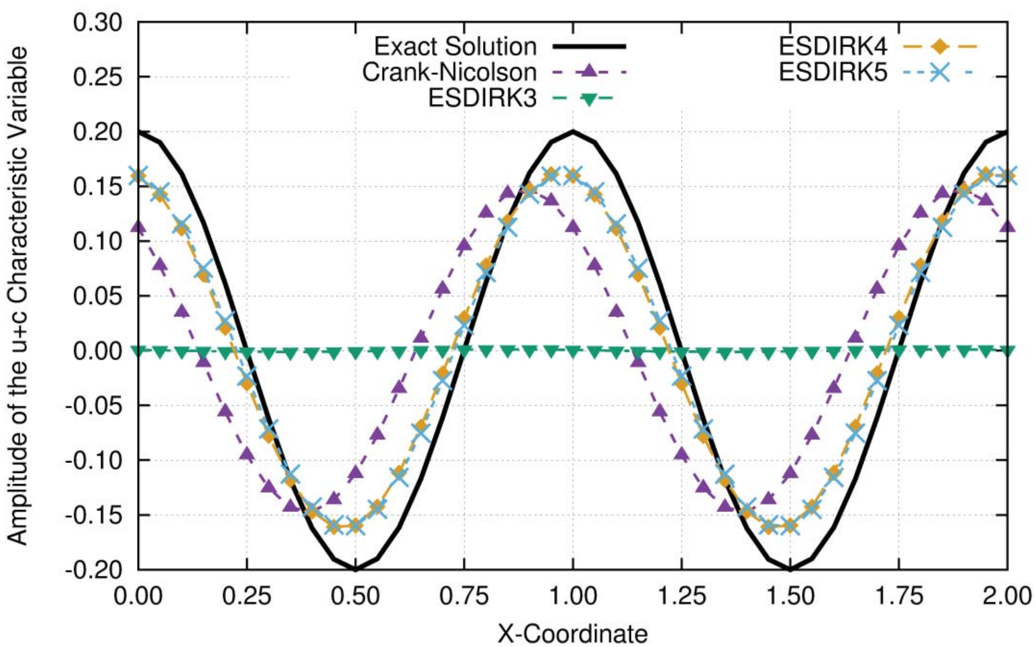
	Dissipation Error		Dispersion Error	
Scheme	VNA	Simulation	VNA	Simulation
Crank-Nicolson	9.02×10^{-5}	2.44×10^{-3}	3.61×10^{-1}	3.61×10^{-1}
ESDIRK3	4.99×10^{-1}	4.90×10^{-1}	1.92×10^{-1}	1.92×10^{-1}
ESDIRK4	7.22×10^{-3}	7.25×10^{-3}	4.90×10^{-2}	4.90×10^{-2}
ESDIRK5	5.10×10^{-2}	5.46×10^{-2}	1.38×10^{-2}	1.39×10^{-2}

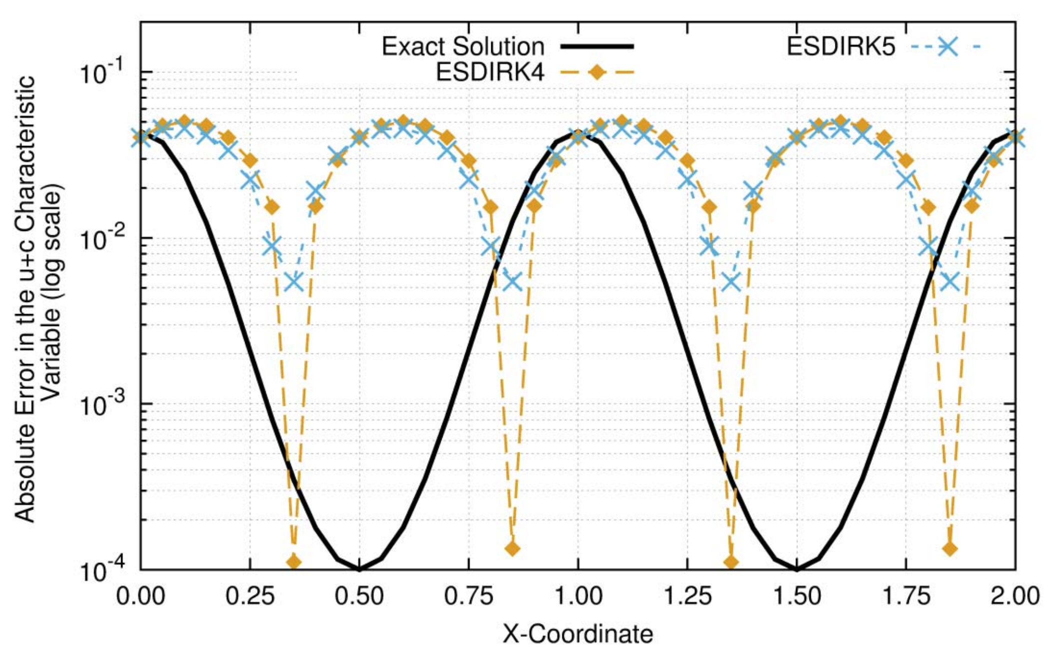




1-D, *CFL = 1.0*, 1000 Periods

	Dissipation Error		Dispersion Error	
Scheme	VNA	Simulation	VNA	Simulation
Crank-Nicolson	2.63×10^{-1}	2.65×10^{-1}	8.11×10^{0}	8.10×10^{0}
ESDIRK3	9.94×10^{-1}	9.94×10^{-1}	1.51×10^{-1}	1.00×10^{-1}
ESDIRK4	2.69×10^{-1}	1.95×10^{-1}	1.50×10^{-2}	3.00×10^{-2}
ESDIRK5	2.70×10^{-1}	2.01×10^{-1}	6.78×10^{-3}	2.50×10^{-2}





3-D Isentropic Vortex

- **Free-stream Mach number of *0.5***

$$\rho_\infty = 1.0\frac{kg}{m^3}, \quad \rho u_\infty = 200.0\frac{kg}{m^2\cdot s}, \quad \rho v_\infty = 0.0\frac{kg}{m^2\cdot s}, \quad \rho w_\infty = 0.0\frac{kg}{m^2\cdot s}, \quad \rho e_{0,\infty} = 305714.3\frac{kg}{m\cdot s^2}$$

$$R_\infty = 287.11\frac{J}{kg\cdot K} \text{ and } \gamma = 1.4$$

- **Perturbation - 11 points across the vortex**

$$\delta u = -\sqrt{R_\infty T_\infty}\frac{\alpha}{2\pi}(y - y_0)\, e^{\phi(1-r^2)}$$

$$\delta v = \sqrt{R_\infty T_\infty}\frac{\alpha}{2\pi}(x - x_0)\, e^{\phi(1-r^2)}$$

$$\delta T = T_\infty \frac{\alpha^2(\gamma - 1)}{16\phi\gamma\pi^2} e^{2\phi(1-r^2)}$$

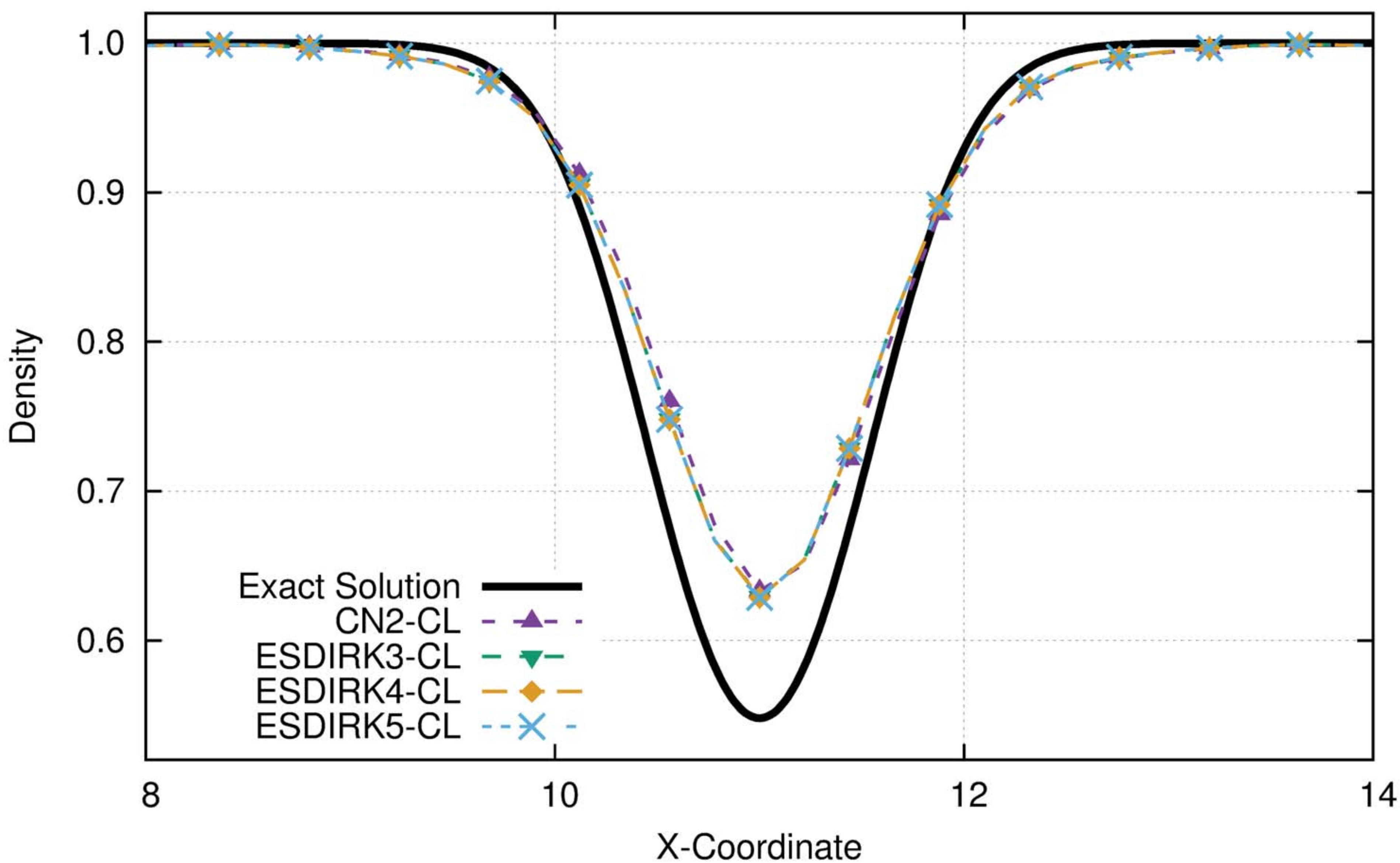
$$\alpha = 4 \text{ and } \phi = 1$$

$$\text{Vortex center: } (x_0, y_0)$$

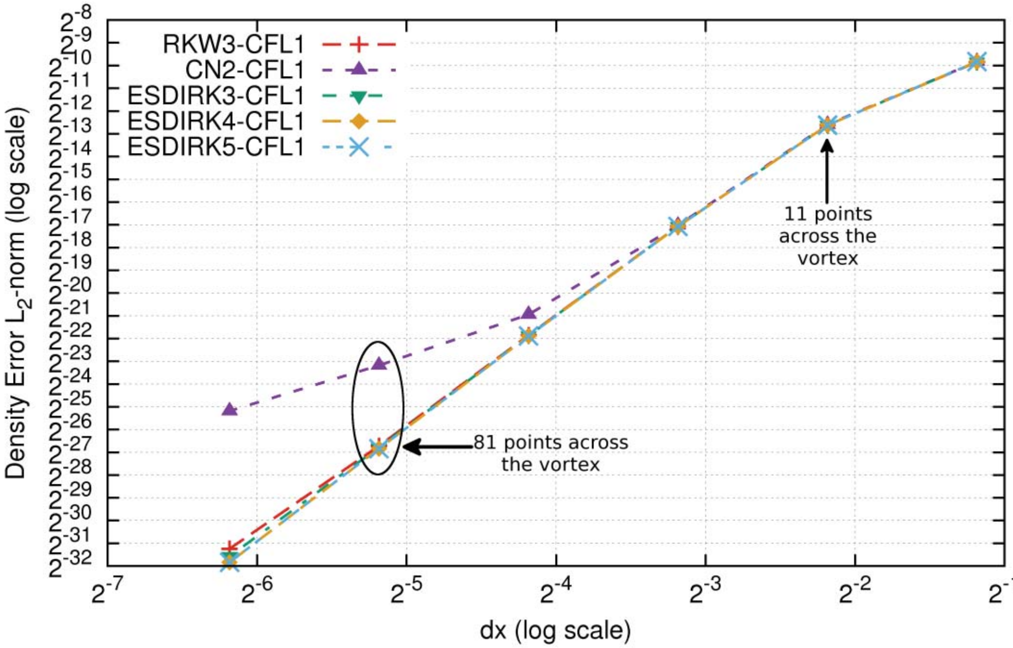
$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

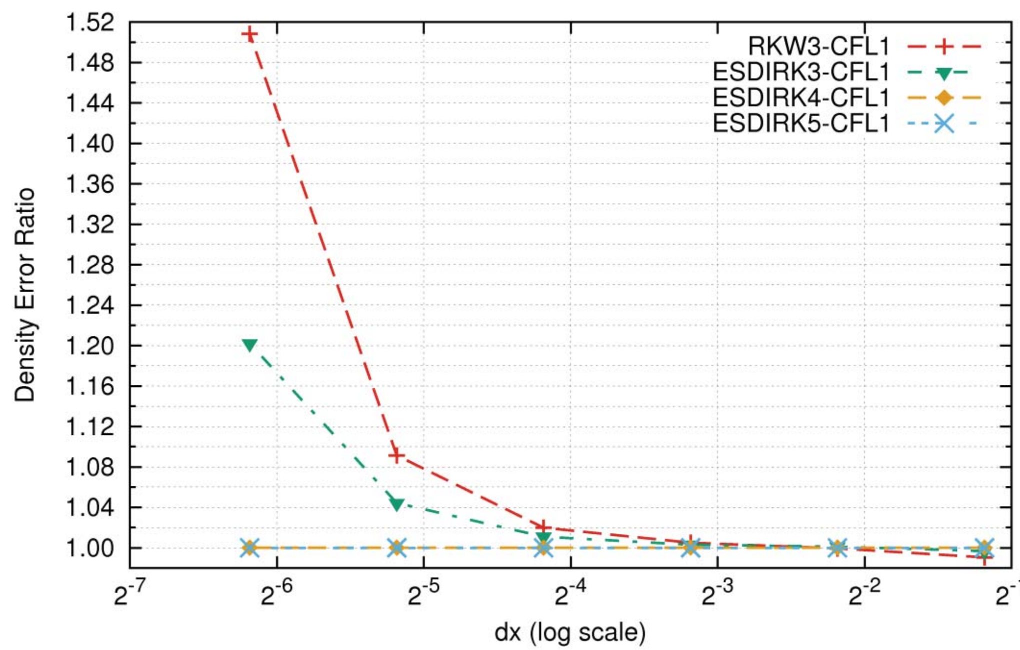
- **More results in the paper**

3-D, *CFL = 1.0*, 40 Lengths, 11 Points Across the Vortex

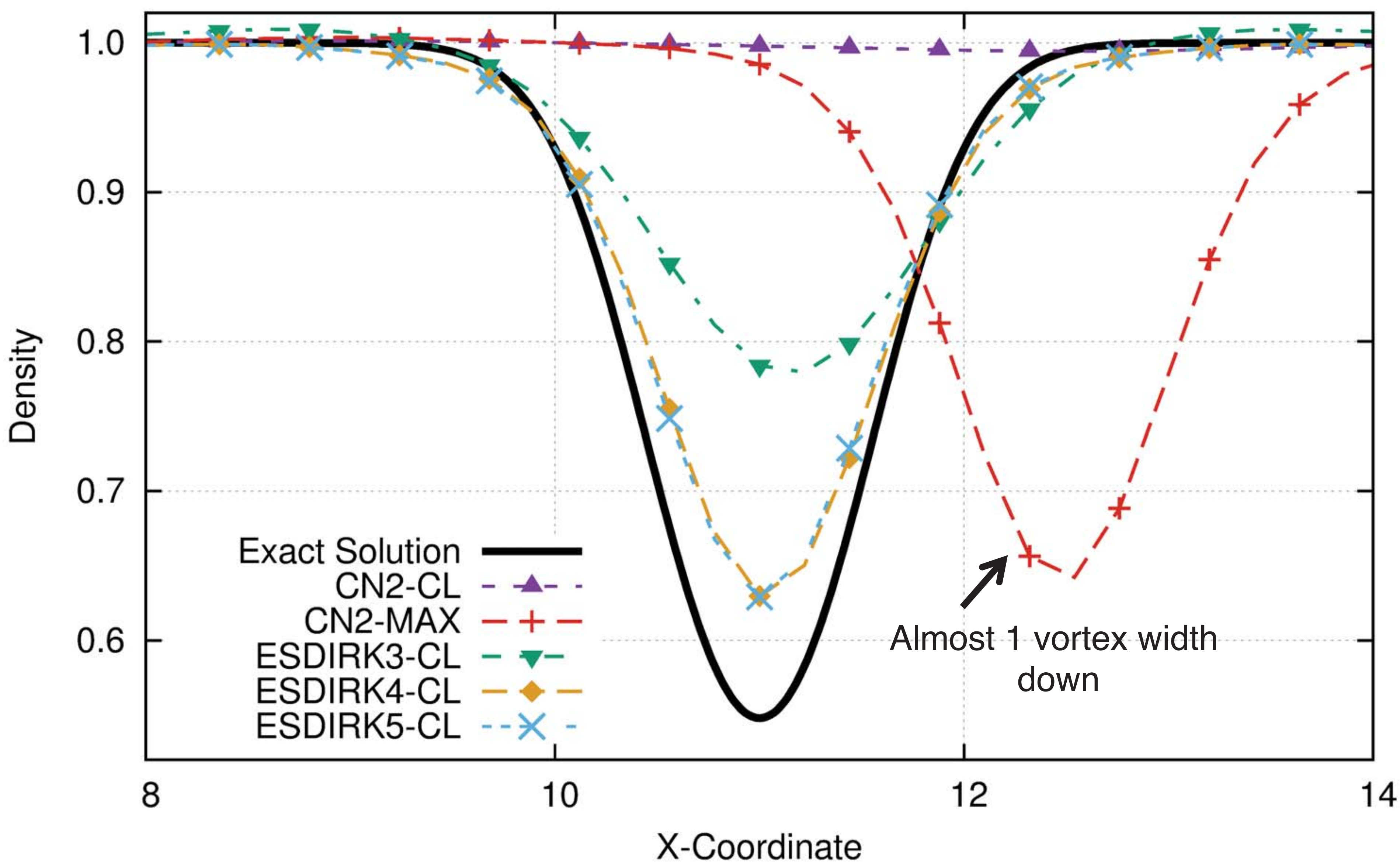


3-D, *CFL = 1.0*
Different Resolutions





3-D, *CFL* = *8.0*, 40 Lengths, 11 Points Across the Vortex

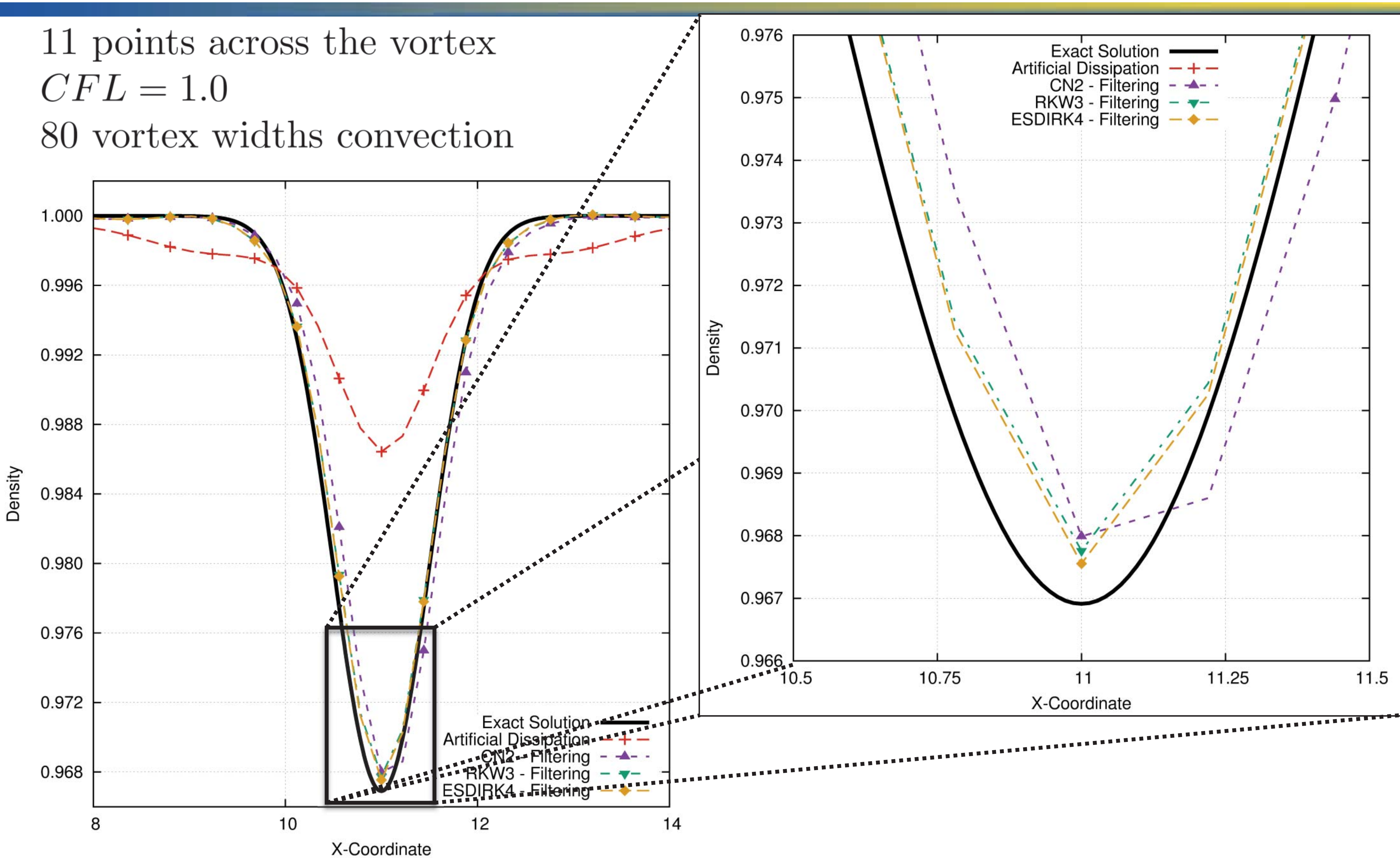


Sneak Peak: Filtering

11 points across the vortex

$CFL = 1.0$

80 vortex widths convection



Conclusions

- ***2nd- and 3rd*-order time integrators for *5th*-order spatial schemes are inadequate**
  - The same order of spatial and temporal discretizations is preferable
  - However, ESDIRK5 is not much better than ESDIRK4
    - 7 implicit stages vs. 5 implicit stages
- **Higher-order time integrators:**
  - Do not show significant improvement on coarse grids at *CFL* of one
  - Are better at high *CFL* number
  - Are better on highly refined grids
- **Spatial error usually dominates for typical *CFL* numbers and grid resolutions**
  - Central difference plus artificial dissipation schemes are inadequate

Future Work

- **Implement more accurate spatial schemes of the same orders of accuracy**
  - Compact-difference schemes
  - Filtering schemes
- **Derive better ESDIRK schemes tailored to the desired dissipation and dispersion properties**
- **Add preconditioning to take maximum advantage of the ESDIRK time integrators for stiff problems**
  - Improved convergence efficiency
  - Improved solution accuracy

Questions???

Extra Slides

3-D, *CFL* = *8.0*
Different Resolutions